

ON THE NUMBER OF NONTRIVIAL PROJECTIVE TRANSFORMATIONS OF CLOSED MANIFOLDS

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ABSTRACT. We show that for a closed Riemannian manifold the quotient of the group of projective transformations by the group of isometries contains at most two elements unless the metric has constant positive sectional curvature or every projective transformation is an affine transformation.

Dedicated to Anatoly Timofeevich Fomenko on his seventieth birthday.

Let (M, g) be a smooth closed Riemannian manifold of dimension $n \geq 2$. By *projective transformation* of (M, g) we understand a diffeomorphism $\phi : M \rightarrow M$ that sends geodesics viewed as unparameterized curves to geodesics. Projective transformations of (M, g) naturally form a group which we denote by Proj . The group of isometries of (M, g) , which we denote by Iso , forms a subgroup of Proj . The following theorem, which is the main result of our paper, answers the following natural question: how big can be Proj/Iso ?

Theorem 1. *Let (M, g) be a smooth connected closed Riemannian manifold of dimension $n \geq 2$. Suppose $|\text{Proj}/\text{Iso}| > 2$. Then, the sectional curvature of g is positive constant or every projective transformation is an affine transformation, i.e., preserves the Levi-Civita connection of g .*

In other words, if a connected closed Riemannian manifold of dimension ≥ 2 whose sectional curvature is not positive constant admits two nonaffine projective transformations ϕ and ψ then the diffeomorphisms $\phi \circ \psi, \psi \circ \phi, \phi^{-1} \circ \psi, \phi \circ \psi^{-1}$ are isometries.

Both possibilities in the conclusion of Theorem 1 can happen. It is well known, see for example [6, Example 2], that for the standard sphere $(S^n, g_{\text{standard}})$ we have

$$\text{Proj}/\text{Iso} = \text{SL}(n+1, \mathbb{R})/\text{SO}(n+1, \mathbb{R})$$

so the set Proj/Iso contains infinitely many elements.

Note though that certain quotients of odd-dimensional spheres of constant positive sectional curvature have $|\text{Proj}/\text{Iso}| = 1$. This follows from [8, Theorem 1], and a concrete 3-dimensional example that can be generalized for all odd dimensions is in [10, §1.4].

Let us also recall an example ([6, Example 4]) such that $|\text{Proj}/\text{Iso}| = \infty$, but each projective transformation is actually an affine transformation. Consider the standard torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, where the action of the group \mathbb{Z}^2 is generated by the standard translations $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+1)$ along the standard basis vectors. The standard flat metric on \mathbb{R}^2 induces a metric on T^2 which we denote by g . Consider now the standard action of $SL(2, \mathbb{Z})$ on \mathbb{R}^2 . It induces a faithful action of $SL(2, \mathbb{Z})$ on T^2 which evidently preserves the Levi-Civita connection of g . Hence, $|\text{Proj}/\text{Iso}| = \infty$. Note that though the example is two-dimensional and flat, it is easy to extend it to nonflat manifolds of higher dimensions by taking direct products with compact manifolds.

Let us now construct an example of a closed Riemannian manifold of arbitrary dimension $n \geq 2$ such that Proj/Iso contains two elements and such that it does not admit affine nonisometric transformations. Two-dimensional version of this example is in [9, §1.3] and in [6, Example 4]. We consider the direct product $S^1 \times M \times S^1$, where S^1 is the circle and M is an arbitrary closed connected manifold of dimension $n-2$. We denote by x and z the standard cyclic coordinates on the first and the second S^1 ; we assume that $x, y \in (\mathbb{R} \bmod 1)$. We will denote by y_1, \dots, y_{n-2} local coordinates on the manifold M . We arbitrary choose a Riemannian metric g on M and a smooth nonconstant 1-periodic function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f > 1$.

Consider the following metric on $S^1 \times M \times S^1$:

$$\left(f(x) - \frac{1}{f(z)}\right) (f(x) - 1) dx^2 + (f(x) - 1) \left(1 - \frac{1}{f(z)}\right) \sum g_{ij} dy^i dy^j + \left(f(x) - \frac{1}{f(z)}\right) \left(1 - \frac{1}{f(z)}\right) dz^2.$$

Next, consider the diffeomorphism $\phi : S^1 \times M \times S^1$ given by $\phi(x, y, z) = (z, y, x)$ (where y denotes a point on M). This pullback with respect to this diffeomorphism is given by

$$\left(f(x) - \frac{1}{f(z)}\right) (f(x) - 1) \frac{f(z)}{f(x)^2} dx^2 + (f(x) - 1) \left(1 - \frac{1}{f(z)}\right) \frac{f(z)}{f(x)} \sum g_{ij} dy^i dy^j + \left(f(x) - \frac{1}{f(z)}\right) \left(1 - \frac{1}{f(z)}\right) \frac{f(z)^2}{f(x)} dz^2,$$

which is projectively equivalent (i.e., has the same geodesics) to the above metric by the Levi-Civita theorem [5], and the diffeomorphism ϕ is therefore a projective transformation.

Remark 1. The diffeomorphism $\phi : (x, y, z) \mapsto (z, y, x)$ used in the example above is not orientable. One can easily modify the example such that the projective transformation is orientable. Indeed, in dimensions ≥ 3 , one can choose (M, g) such that it admits an orientation-reversing isometry $\alpha : M \rightarrow M$, and instead of diffeomorphism ϕ consider the diffeomorphism $(x, y, z) \mapsto (z, \alpha(y), x)$. Essentially the same idea works in dimension 2 as well: one takes the function f such that it is even and superpose the diffeomorphism ϕ with an orientation-reversing isometry $(x, y, z) \mapsto (-x, y, z)$.

The following special case of Theorem 1 is due to A. Zeghib [11], where also some previous results in this direction are listed, see also the introductions to [9] and to [6] for an overview and for the history of the problem. Zeghib [11, Theorem 1.3] has proved Theorem 1 under a stronger assumption $|\text{Proj}/\text{Iso}| > 2n$. Actually, our proof follows the lines of the Zeghib's proof and is based on his results and ideas; we will clearly explain the additional argument that allowed us to improve his result.

Remark 2. Theorem 1 remains correct if one replaces closeness (of our manifold (M, g)) by completeness. This improved statement is based on certain nontrivial ideas and calculations that are invented in [1] and will be published after or in that paper.

Proof of Theorem 1. Within the proof we assume that (M, g) is a closed connected Riemannian manifold of dimension at least 2 admitting at least one projective transformation which is not an affine transformation. We also assume that the sectional curvature is not positive constant. Our goal is to show $|\text{Proj}/\text{Iso}| \leq 2$.

Consider the metrization equation from [2, Theorem 2.2]. The precise formula of this equation is not important for us and its introducing requires work we do not want to invest, we refer to [2] for details. We will list here the properties of this equation and its solutions which will be used in the proof.

- (I) The metrization equation is a (homogeneous) linear system of PDE, so its solution space which we denote by Sol is a linear vector space. By [7, Theorem 2] (or alternatively [9, Theorem 16], [3, Theorem 1], [10, Corollary 5.2]; the two-dimensional version follows from [4]), under our assumptions, $\dim \text{Sol} \leq 2$.
- (II) In a local coordinate system, the solutions of the metrization equation can be viewed by matrices¹ whose components are functions of the coordinates. Nondegenerate (i.e., with nowhere vanishing determinant) solutions σ^{ij} correspond, in a local coordinate system, to metrics (of arbitrary signature) projectively equivalent to g (i.e., having the same geodesics with g). The correspondence is given by the formula

$$(1) \quad \sigma^{ij} = g^{-1} |\det g|^{\frac{1}{n+1}} \quad \text{and} \quad g^{-1} := \det |\sigma| \sigma^{ij}.$$

From the formula we clearly see that positively definite σ^{ij} correspond to positively definite, i.e., Riemannian, metrics.

- (III) Metrization equations are projectively invariant, so for any projective transformation ϕ the pullback $\phi^* \sigma$ of a solution is a solution.

¹As geometric objects they are weighted symmetric $(2, 0)$ -tensors; in particular their pullback is well-defined.

(IV) If a solution σ is nondegenerate at every point, then $\sigma^{-1}\bar{\sigma} := (\sigma^{-1})_{is}\bar{\sigma}^{sj}$ (where $\bar{\sigma}$ is also a solution), is a well-defined (1,1)-tensor field. If σ and $\bar{\sigma}$ correspond to the metrics g and \bar{g}

by the formula (1), we have $\sigma^{-1}\bar{\sigma} = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{js} g_{si}$.

Suppose ϕ is a projective transformations. Take a basis $\sigma, \bar{\sigma}$ in Sol and consider the pullbacks $\phi^*\sigma, \phi^*\bar{\sigma}$. They also belong to Sol and are therefore linear combinations of the basis solutions σ and $\bar{\sigma}$; we denote the coefficients as below:

$$\begin{pmatrix} \phi^*\sigma \\ \phi^*\bar{\sigma} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma \\ \bar{\sigma} \end{pmatrix} = \begin{pmatrix} a\sigma & + & b\bar{\sigma} \\ c\sigma & + & d\bar{\sigma} \end{pmatrix}.$$

We denote the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ above by A or by A_ϕ . The mapping from Proj to $GL(2, \mathbb{R})$ given by $\phi \mapsto A_\phi$ is actually a representation and if A_ϕ is the identity matrix then ϕ is an isometry. The composition $\psi \circ \phi$ of two projective transformations corresponds to the product of matrices A_ψ and A_ϕ is the reverse order:

$$\psi \circ \phi \mapsto A_\phi A_\psi.$$

In [11] it was shown that, in our assumptions, if A_ϕ has real eigenvalues, then ϕ is an isometry. The result is nontrivial and of huge importance for us since further we may assume that the matrices A_ϕ have nonreal eigenvalues.

Let us now consider the case when the matrix $A = A_\phi$ has complex-conjugate nonreal eigenvalues. Depending on the sign of the determinant, by the choice of the basis in Sol, we may therefore assume that the matrix A is as in one of the following two cases:

$$A = C \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \quad \text{or} \quad A = C \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}$$

with $C > 0$. The first case is when $\det A > 0$ and the second is when $\det A < 0$. Moreover, without loss of generality we may assume in the first case that the basis $\sigma, \bar{\sigma}$ is such that the metric g corresponds to the basis solution σ .

Let us show that the first case is impossible, unless $C = 1$ and α is such that $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is precisely the argument overseen by Zeghib in his paper.

In order to do this, let us take an arbitrary point $p \in M$ and a basis in $T_p M$ such that

$$\sigma = \text{diag}(1, \dots, 1) \quad \text{and} \quad \bar{\sigma} = \text{diag}(s_1, \dots, s_n).$$

The existence of such a basis is trivial since g is positively definite.

Next, consider

$$\phi^*\sigma, \quad \phi \circ \phi^*\sigma = \phi^*(\phi^*(\sigma)), \quad \phi \circ \phi \circ \phi^*\sigma = \phi^*(\phi^*(\phi^*(\sigma))), \dots, \underbrace{\phi \circ \dots \circ \phi^*}_{k \text{ times}} \sigma, \dots$$

Since the matrix corresponding to the superposition $\underbrace{\phi \circ \dots \circ \phi}_{k \text{ times}}$ is $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^k = \begin{pmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{pmatrix}$,

we have that at our point p the matrix $\sigma^{-1} \left(\underbrace{\phi \circ \dots \circ \phi^*}_{k \text{ times}} \sigma \right)$ is

$$(2) \quad \sigma^{-1}(C^k(\cos k\alpha\sigma + \sin k\alpha\bar{\sigma})) = C^k \text{diag}(\cos k\alpha + s_1 \sin k\alpha, \dots, \cos k\alpha + s_n \sin k\alpha).$$

We will need the following simple lemma:

Lemma 1. *Suppose for all $k \in \mathbb{N}$ we have $\cos k\alpha + s \sin k\alpha > 0$. Then, α is such that*

$$\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof of Lemma 1. First observe that $\cos k\alpha + s \sin k\alpha > 0$ is the 1st coordinate of the $k\alpha$ -rotation of the vector $\begin{pmatrix} 1 \\ s \end{pmatrix}$ around the origin. If $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the set of the points of \mathbb{R}^2 of the form $\begin{pmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{pmatrix} \begin{pmatrix} 1 \\ s \end{pmatrix}$ is either dense on a circle centered at the origin, or coincides with the set of vertices of a regular (may be degenerate, i.e., containing only two vertices) polygon containing the origin. In both case there exists a point such that its first coordinate is nonpositive and we get a contradiction. **Lemma 1 is proved.**

We now continue the proof of Theorem 1. Combining Lemma 1 with (2), we see that if $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, there exists k such that the (diagonal) matrix of $\sigma^{-1}(\phi \circ \dots \circ \phi^* \sigma)$ has a nonpositive eigenvalue. But since σ corresponds to the metric and is therefore positively definite at all points of the manifold which implies of course that its pullback is also positively definite, this is impossible. Thus, $A_\phi = C \text{Id}$. Since closed manifolds do not admit nontrivial homotheties, $C = 1$ and ϕ is an isometry.

Finally, we obtained that for every projective transformation ϕ such that it is not an isometry the matrix A_ϕ has two complex conjugate nontrivial eigenvalues and negative determinant. Then, for two such (nonisometric) projective transformations ϕ and ψ there superpositions $\phi \circ \psi$ is an isometry, since the product of two matrices A_ψ and A_ϕ with negative determinants has positive determinant. Then, all projective transformations such that they are not isometries lie in the same equivalence class of Proj/Iso which implies that the number of elements in Proj/Iso is at most 2. **Theorem 1 is proved.**

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